Distance metric to $\pi(X)$.

Kathryn, 2/16/18

Question A. The idea is that we score solutions, based in part on their proximity to $\pi(X)$. How should we choose what proximity means, i.e. which metric to use in $\mathbb{R}^3$?

Question B. What error metric should we use on the space of varieties? Or rather: given an approximation $\hat{g}$ to the Gr obner basis g, obtained via regression and monomial selection or via a $\mathbb{Q}$ approximation of a $\mathbb{R}$ g, how "good" of an approximation to $V(g)$ is $V(\hat{g})$? We could use Euclidean distance from $\pi(X)$, but it's nonuniform with respect to shape space. This is important because we're not really comparing the distance between pairs of points, but rather between pairs of shapelets. Therefore we opt for a metric based on the geodesic distance in shape space [KvD]. That is, given samples $\{\mathbf{k}^{(1)}, \ldots, \mathbf{k}^{(N)}\} \subset V(\hat{g})$,

$$\varepsilon(g, \hat{g}) := \max_{1 \leq j \leq N} \left\{ \min_{\mathbf{v} \in V(g)} d(\mathbf{v}, \mathbf{k}^{(j)})^2 \right\}$$

where d is the geodesic distance between two shapelets.

(0) Let $(c, d, e) = \mathbf{v}$. Let $(c', d', e') = \mathbf{k}^{(j)}$.

(1) We need to put $f_{xx}, f_{xy}, f_{yy} \to t, r, s$. So for us that's

$$r = \frac{1}{2}(c - e), \quad s = d, \quad t = \frac{1}{2}(c + e) \qquad \Longrightarrow \qquad \mathbf{x} = (r, s, t).$$

$$r' = \frac{1}{2}(c' - e'), \quad s' = d', \quad t' = \frac{1}{2}(c' + e') \qquad \Longrightarrow \qquad \mathbf{x}' = (r', s', t').$$

We can also express this as $\mathbf{x} = A\mathbf{v}$ and $\mathbf{x}' = A\mathbf{k}^{(j)}$, where $A := \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)$.

(2) We need to rotate t, r, s. Let $\mathbf{n} := \mathbf{x} \times \mathbf{x}'$.

$$R := \begin{pmatrix} \frac{\mathbf{x}}{||\mathbf{x}||} & \frac{\mathbf{x}\times\mathbf{n}}{||\mathbf{x}\times\mathbf{n}||} \end{pmatrix}^T = \begin{pmatrix} \frac{r}{\sqrt{r^2+s^2+t^2}} & \frac{-r's^2+rs's-r't^2+rtt'}{\sqrt{(-t'r^2+r'tr+ss't-s^2t')^2+(-r's^2+rs's-r't^2+rtt')^2+(-s'r^2+r'sr-s't^2+stt')^2}} \\ \frac{s}{\sqrt{r^2+s^2+t^2}} & \frac{-s'r^2+r'sr-s't^2+stt'}{\sqrt{(-t'r^2+r'tr+ss't-s^2t')^2+(-r's^2+rs's-r't^2+rtt')^2+(-s'r^2+r'sr-s't^2+stt')^2}} \\ \frac{t}{\sqrt{r^2+s^2+t^2}} & \frac{-t'r^2+r'tr+ss't-s^2t'}{\sqrt{(-t'r^2+r'tr+ss't-s^2t')^2+(-r's^2+rs's-r't^2+rtt')^2+(-s'r^2+r'sr-s't^2+stt')^2}} \end{pmatrix}^T$$

$$\mathbf{y} = R\mathbf{x} = \begin{pmatrix} \frac{r^2+s^2+t^2}{\sqrt{r^2+s^2+t^2}} \\ \frac{rst'-rs't+sr't-srt'+trs'-tr's}{\sqrt{r^2(s'^2+t'^2)-2rr'(ss'+tt')+r'^2(s^2+t^2)+(s't-st')^2}} \end{pmatrix} = \begin{pmatrix} \sqrt{r^2+s^2+t^2} \\ 0 \end{pmatrix}$$

$$\mathbf{y}' = R\mathbf{x}' = \begin{pmatrix} \frac{rr'+ss'+tt'}{\sqrt{r^2+s^2+t^2}} \\ -\frac{\sqrt{(r^2+s^2+t^2)(r^2(s'^2+t'^2)-2rr'(ss'+tt')+r'^2(s^2+t^2)+(s't-st')^2)}}{r^2+s^2+t^2} \end{pmatrix} = \begin{pmatrix} \sqrt{r'^2+s'^2+t'^2} \\ 0 \end{pmatrix}$$

(3) Put $R(\mathbf{x}) \to$ polars. Rule $\rho = ||\mathbf{y}||$, $\phi = \arctan(\mathbf{y}_2/\mathbf{y}_1)$.

$$\rho = \sqrt{r^2+s^2+t^2} = C$$

$$\phi = 0$$

$$\rho' = \sqrt{r'^2+s'^2+t'^2} = C'$$

$$\phi' = -\arctan\left(\frac{\sqrt{(r^2\left(s'^2+t'^2\right)-2rr'(ss'+tt')+r'^2\left(s^2+t^2\right)+(s't-st')^2)}}{rr'+ss'+tt'}\right)$$

Proposition 0.1: From Notes_Feb9

The geodesic squared-distance between $\mathbf{v}, \mathbf{k} \neq 0$ is

$$d(\mathbf{v},\mathbf{k})^2 = (\rho-\rho')^2\left(\left(\frac{\phi-\phi'}{\ln\left(\frac{\rho}{\rho'}\right)}\right)^2+1\right)$$

where $RA\mathbf{v} \to (\rho,\phi)$ and $RA\mathbf{k} \to (\rho',\phi')$.

This makes

$$\begin{aligned} d(\mathbf{v},\mathbf{k})^2 &= \left(C-C'\right)^2\left(\left(\frac{\arctan\left(\frac{\sqrt{\left(r^2\left(s'^2+t'^2\right)-2rr'(ss'+tt')+r'^2\left(s^2+t^2\right)+(s't-st')^2\right)}}{rr'+ss'+tt'}\right)}{\ln\left(\frac{C}{C'}\right)}\right)^2+1\right) \\ &= \left(\frac{\sqrt{c^2+2d^2+e^2}}{\sqrt{2}}-\frac{\sqrt{c'^2+2d'^2+e'^2}}{\sqrt{2}}\right)^2\left(\left(\frac{\arctan\left(\frac{\sqrt{\left(c^2\left(2d'^2+e'^2\right)-2cc'(2dd'+ee')+c'^2\left(2d^2+e^2\right)+2(d'e-de')^2\right)}}{cc'+2dd'+ee'}\right)}{\frac{1}{2}\log\left(\frac{c^2+2d^2+e^2}{c'^2+2d'^2+e'^2}\right)}\right)^2+1\right) \\ &= \frac{1}{2}\left(\sqrt{c^2+2d^2+e^2}-\sqrt{c'^2+2d'^2+e'^2}\right)^2\left(\left(\frac{2\arctan\left(\frac{\sqrt{\left(c^2\left(2d'^2+e'^2\right)-2cc'(2dd'+ee')+c'^2\left(2d^2+e^2\right)+2(d'e-de')^2\right)}}{cc'+2dd'+ee'}\right)}{\log\left(\frac{c^2+2d^2+e^2}{c'^2+2d'^2+e'^2}\right)}\right)^2+1\right) \end{aligned}$$

Let's examine[1] the surflets precisely distance k away from a fixed surface $(1,1,1)$. Displaying only $\varphi = 0$, i.e. requiring $\frac{1}{2}\arctan\left(\frac{2d}{c-e}\right) = 0 \iff d = 0$. Thus the 1-sphere[2] centered at the base surflet $\mathbf{cde}_{base} := (c_{base}, d_{base}, e_{base})$ of radius $\in \mathbb{R}_+$ is given by

$$S_{base}(k) := \left\{\mathbf{cde} = (c,d,e) \in \mathbb{R}^3 : d(\mathbf{cde}, \mathbf{cde}_{base}) = k \text{ and } d = 0\right\}.$$

Example 1. *$S_{(1,0,1)}(k)$ is the k-level set of*

$$\begin{aligned} F_{(1,0,1)}(c,e) &:= d((c,0,e),(1,0,1)) \\ &= \frac{2\left(2-\sqrt{c^2+e^2}\right)^2\left((0.25\ln\left(c^2+e^2\right) - 0.693147)\ln\left(c^2+e^2\right) + \arctan\left(\frac{e}{c}\right)^2 + 0.480453\right)}{\left(\ln\left(c^2+e^2\right) - 1.38629\right)^2} \end{aligned}$$

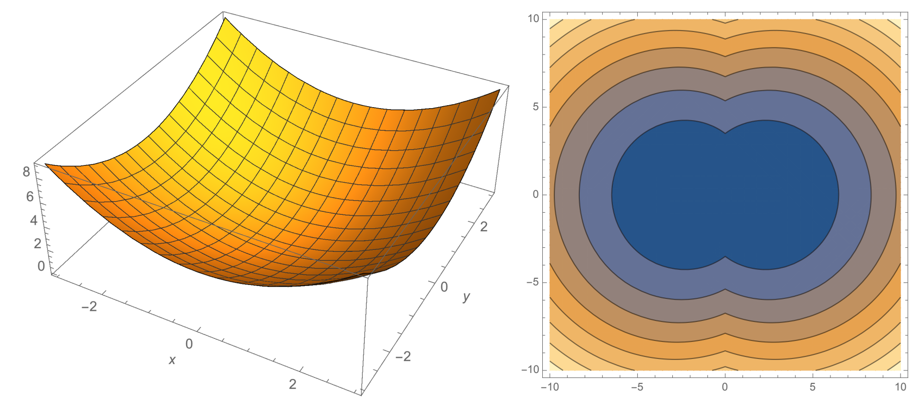


***Figure**: Left, our base surflet $(1,0,1)$. Right, [[Outdated. See below.]]the level sets of $F_{(1,0,1)}$; this will depend on what value at which you fix d. Every point on the k-level set of F corresponds to a surflet that is distance k from surflet $(1,0,1)$.*

We'll sample a few points from $S_{(1,0,1)}(4)$. Take $e = 0$. Then we'll find $c > 0$ that satisfies

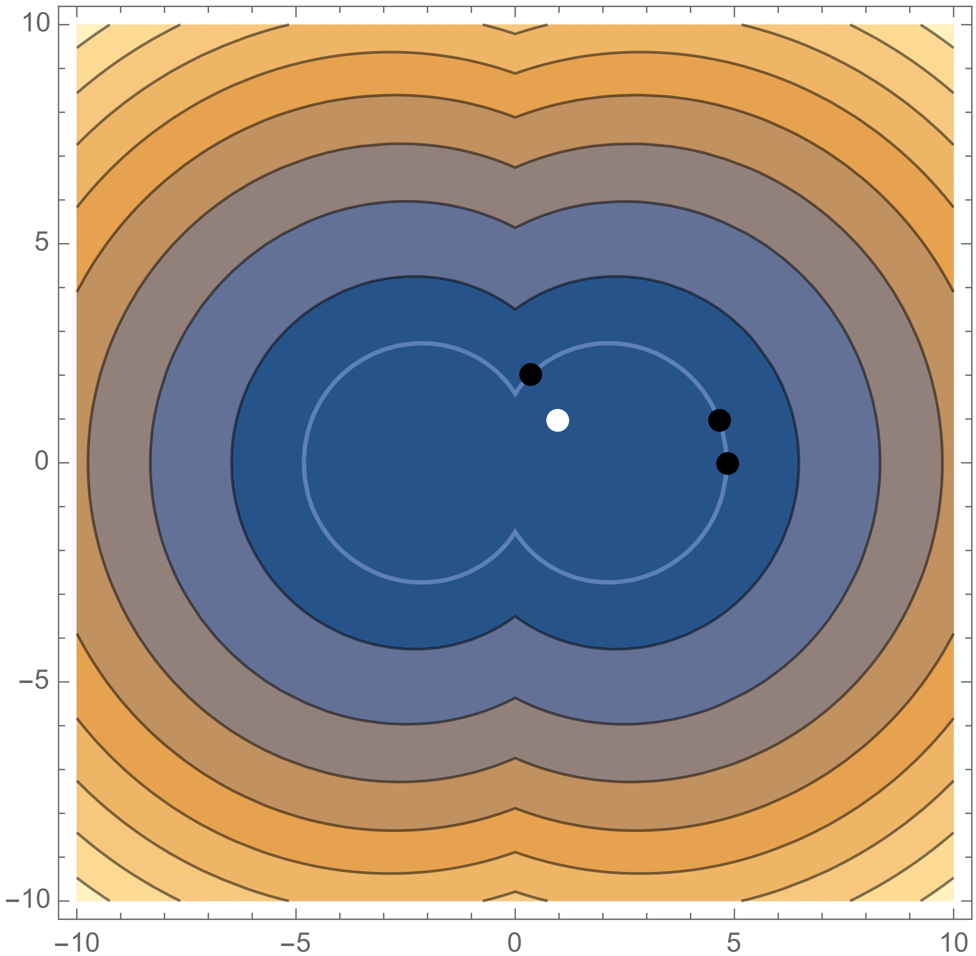
$$4 == F_{(1,0,1)}(c,0) = \frac{2(2-c)^2\left((0.5\ln(c) - 0.693147)\,2\ln(c) + 0.480453\right)}{\left(2\ln(c) - 1.38629\right)^2}$$

$$2\left(2\ln(c) - 1.38629\right)^2 = (2-c)^2\left((0.5\ln(c) - 0.693147)\,2\ln(c) + 0.480453\right)$$

From inspection of the level set, we see that $c = 4.83, e = 0$ is an approximate *solution. This is the degenerate surflet*

[1] GeoDist.nb

[2] or some homotopy equivalent 1-simplex



Issue to fix: the white dot should be in the center of the contours.

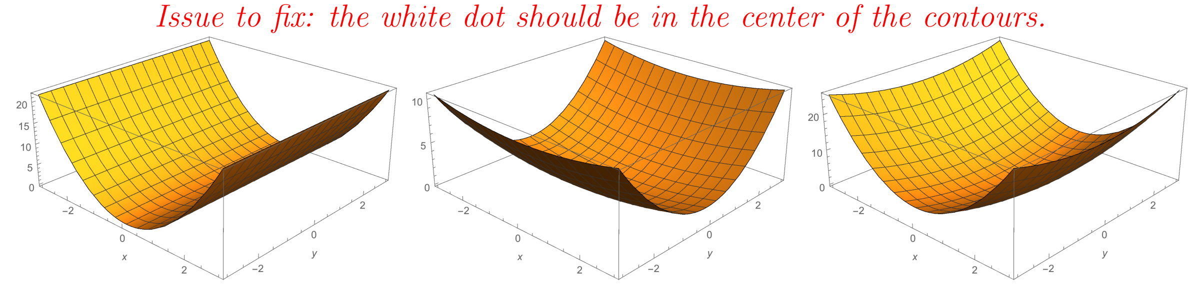


Figure*: Surflets of geodesic distance approximately 4 from the base surflet. Top: the positions of the surflets (black points) with respect to the level set* $F = 4$ *(light blue curve) and the base surflet (white point).* $Left : (4.83, 0, 0)$, $Center : (0.335, 0, 2)$, $Right : (4.644, 0, 1)$.

It does this because $F \to \infty$ as $\mathbf{cde} \to \mathbf{cde}_{base}$. Notice that $\left(\frac{C-C'}{\ln(C/C')}\right)^2$ has some bizarre blow-up around $C = C'$ (see figure).

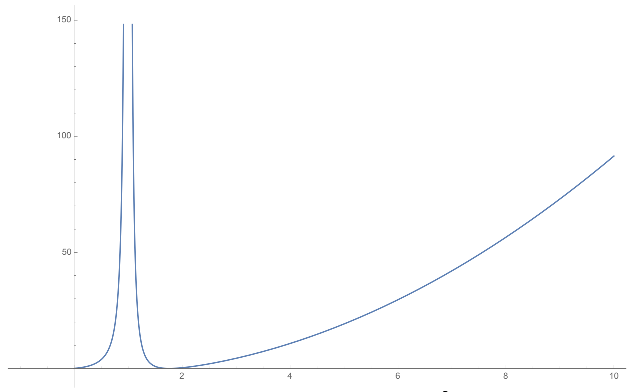


Figure: $\left(\frac{C-C'}{\ln(C/C')}\right)^2$

Important thing we still need to do: show that WLOG we can fix $d = 0$.

To sample the k-level set starting at the base surflet: Choose some b at random (not referring to first-order parameter right now). Note: b is random but still the log spiral MUST PASS through the base, so let $\rho_0 = a_0 e^{b\phi_0} \iff a_0 := \rho_0 e^{-b\phi_0}$. Then you'll get a random logarithmic spiral passing through the origin and the base surflet, that is, $\rho = a_0 e^{b\phi}$. Along this path, the distance to the base surflet will be minimal at the basepoint, and will behave nicely (unlike the figure above). [3] Searching along this spiral, find the 2 points for which $d = k$.

Procedure 1 (Sampling surflets from geodesic distance k away from the base)**.** *Fix an orientation*[4] $\varphi \sim d \sim s$*, and a base surflet with polar coordinates* (ρ_0, ϕ_0)*. Choose a random* $\alpha \in \mathbb{R}$*. Then* (ρ, ϕ) *defines a map from* α *to a surflet*[5] *with polar coordinates* $\phi(\alpha) = \phi_0 + (\ln|\beta|)/\alpha$ *and* $\rho(\alpha) = \rho_0\beta$*, where*

$$\beta = 1 + \frac{k\alpha}{\rho_0\sqrt{1+\alpha^2}}.$$

This surflet is of geodesic distance k *away from the base.*

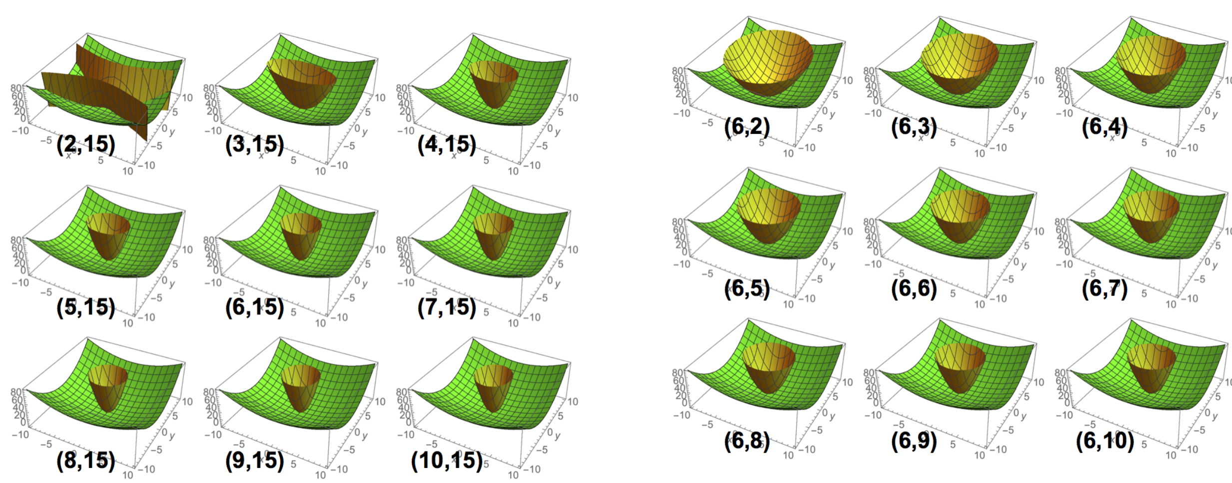


Figure: Surflets centered at the base surflet $(c, d, e) = (1, 0, 2)$. Coordinates given are (α, k).

[3] So, not *lines* through the base surflet, but rather log *spirals* through the base surflet.

[4] I think we can say WLOG $s = 0$...? This procedure requires it I think.

[5] Truly it's a *pair* of surflets, with $\beta = 1 \pm k\alpha/\rho_0\sqrt{1+\alpha^2}$, but the other one was a hassle.

Proof of Procedure 1. Given random b, and the base surflet's polar coordinates ρ_0, ϕ_0, we simply substitute $\phi(\alpha)$ and $\rho(\alpha)$ into the geodesic distance function.

$$d([\rho(\alpha), \phi(\alpha)], [\rho_0, \phi_0])^2 = \left(\rho_0\left(1 \pm \frac{k\alpha}{\rho_0\sqrt{1+\alpha^2}}\right) - \rho_0\right)^2 \left(\left(\frac{\phi_0 + \frac{1}{\alpha}\ln\beta - \phi_0}{\ln\left(\frac{\rho_0\beta}{\rho_0}\right)}\right)^2 + 1\right)$$
$$= \rho_0^2\left(\frac{k\alpha}{\rho_0\sqrt{1+\alpha^2}}\right)^2 \left(\left(\frac{\frac{1}{\alpha}\ln\beta}{\ln\beta}\right)^2 + 1\right) = \left(\frac{k}{\sqrt{1+\alpha^2}}\right)^2 \left(\left(\frac{\ln\beta}{\ln\beta}\right)^2 + \alpha^2\right)$$
$$= k^2\frac{1+\alpha^2}{1+\alpha^2} = k^2.$$

$\square$

Example 2. *When $s = 0$, $C = \sqrt{r^2+t^2} = \sqrt{(c^2+e^2)/2}$, $\sigma = \arctan(t/r) = \arctan((c+e)/(c-e))$, and $\varphi = \arctan(0)/2 = 0$, so $(r,t) \to (C,\sigma)$ is simply a change from Cartesian to polar coordinates. Therefore $\rho = C$ and $\phi = \sigma$. We then have $\sigma(\alpha) = \sigma_0 + (\ln\beta)/\alpha$ and $C(\alpha) = C_0\beta$, where*

$$\beta = 1 \pm \frac{k\alpha\sqrt{2}}{\sqrt{c_0^2+e_0^2}\sqrt{1+\alpha^2}}.$$

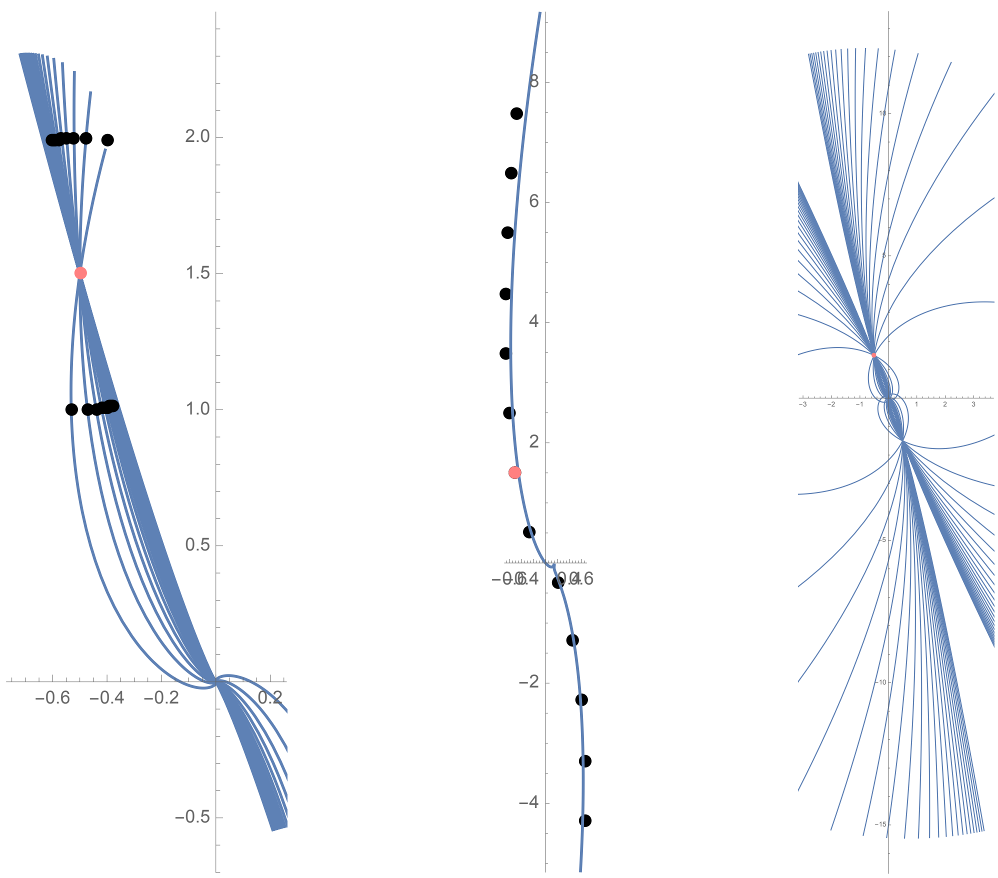


Figure: Interesting families of surflets (black points). Consider $s = 0$. **Left**, the family of log spirals as α varies, and points at level set $k = 1/2$. **Center**, the family of log spirals as k varies, and points at fixed $\alpha = 6$. Pink points denote base surflet in (r,t) coordinates. **Right**, the family of log spirals for base surflet $(r,s,t) = (-1/2, 0, 3/2)$ in pink, as $-22 \le \alpha \le 22$ and

$-14 \leq k \leq 14$. Note that for some values of (α, k) in this range (i.e. at zero) there are no corresponding spirals.

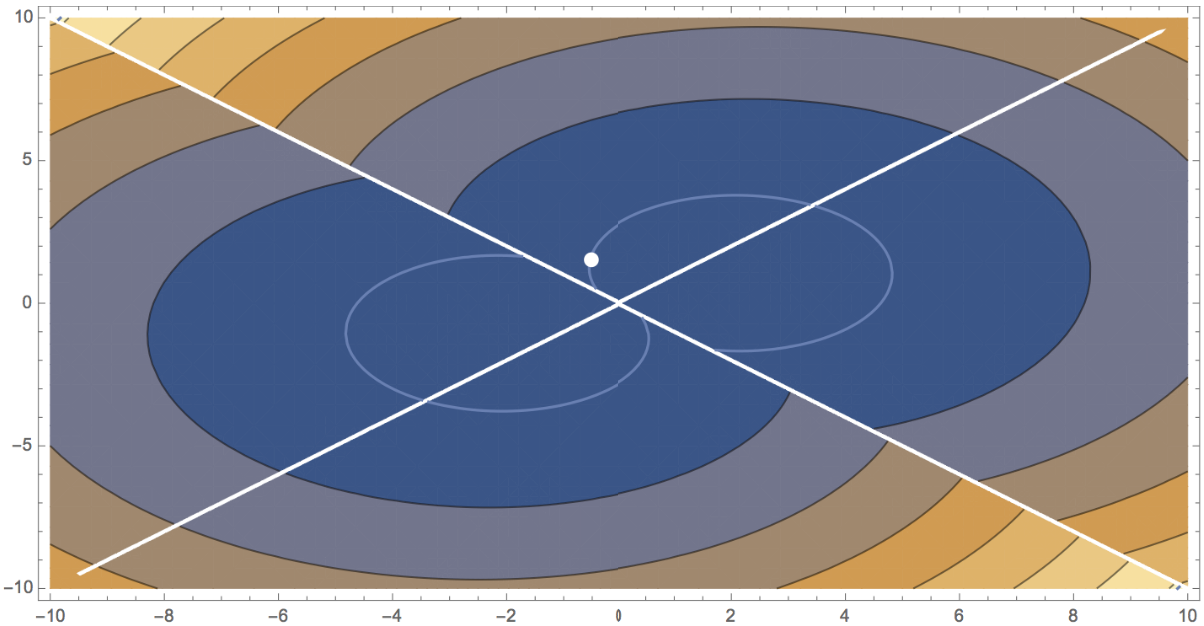


Figure: Updated level sets (wouldn't render as one so I had to plot it in halves) of the geodesic to the base surflet (white point). Axes are c, e, with $d = 0$ fixed. Note the visible $c = e$ and $c = -e$ asymptotes.

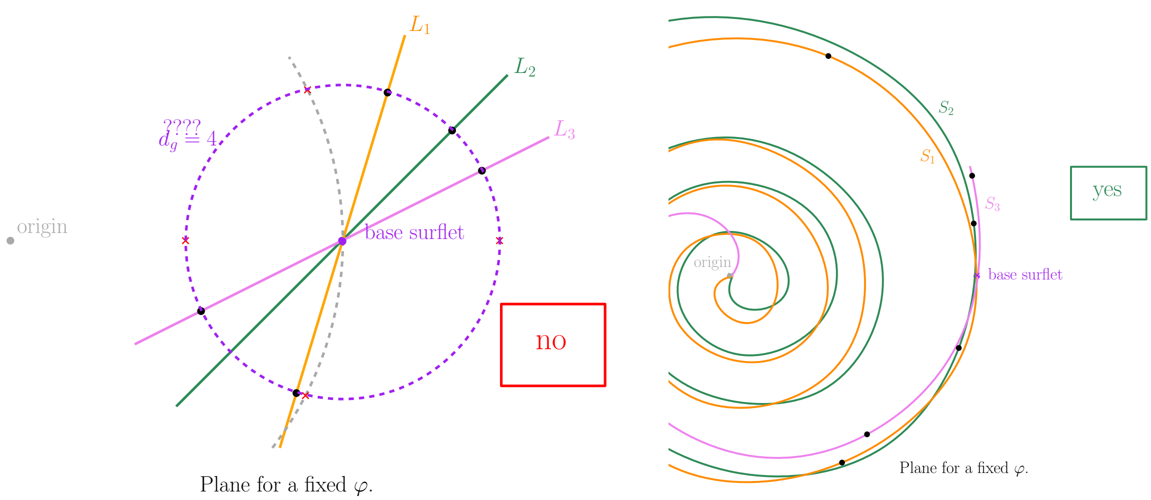


Figure: Left, using lines. Right, using spirals.

Derivation of Procedure 1: Given random α, and a_0 as defined above, and the base surflet ρ_0, ϕ_0, solve

$$k = \frac{a_0\sqrt{1+\alpha^2}}{\alpha}|e^{\alpha\phi} - e^{\alpha\phi_0}| = \frac{\rho_0 e^{-\alpha\phi_0}\sqrt{1+\alpha^2}}{\alpha}|e^{\alpha\phi} - e^{\alpha\phi_0}| = \frac{\rho_0\sqrt{1+\alpha^2}}{\alpha}|e^{\alpha(\phi-\phi_0)} - 1|$$

for ϕ. Then

$$k^2 = \frac{a^2(1+\alpha^2)}{\alpha^2}(e^{\alpha\phi} - e^{\alpha\phi_0})^2 \implies \frac{k^2\alpha^2}{a_0^2(1+\alpha^2)} = (e^{\alpha\phi} - e^{\alpha\phi_0})^2 \implies e^{\alpha\phi_0} \pm \frac{k\alpha}{a_0\sqrt{1+\alpha^2}} = e^{\alpha\phi}$$

$$\phi = \frac{1}{\alpha}\ln\left(e^{\alpha\phi_0} \pm \frac{k\alpha}{a_0\sqrt{1+\alpha^2}}\right) = \frac{1}{\alpha}\ln\left(e^{\alpha\phi_0} \pm \frac{k\alpha e^{\alpha\phi_0}}{\rho_0\sqrt{1+\alpha^2}}\right) = \phi_0 + \frac{1}{\alpha}\ln\beta$$

which then means

$$\rho = a_0 e^{\alpha\phi} = a_0 e^{\alpha\phi_0+\ln\beta} = \rho_0 e^{-\alpha\phi_0} e^{\alpha\phi_0} e^{\ln\beta} = \rho_0\beta.$$