231A Final:
Variations on the Nerve Theorem

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Contents

1 Foreword

Given finitely many points sampled from a smooth manifold (in application, this is usually assumed to be Riemannian) E, one may be interested in "approximating" this as-of-yet unknown surface E. The goal is generally to identify up to isomorphism the homology groups, and if possible, the homotopy groups, of E. One obstacle to obtaining these algebraic descriptions of E is that the surface may be very complicated. A currently studied method is to replace E with a covering of convex, closed sets, and to identify the homology and homotopy groups of the *nerve* of that cover. This application is our motivation to study the relationship of the homotopy types of the space E, a given open cover $\mathcal{U}$, and the nerve $N(\mathcal{U})$ of that cover.

Before introducing terminology and defining key terms, we will state the main results of each of the four featured authors. These statements (presented in their original language) will reappear as highlighted theorems in the later sections. Assume the ambient space (i.e. the manifold that holds submanifold E) is Hausdorff.

Definition 1 (Notation)**.** *In this study, $\simeq$ will denote homotopy equivalence.*

Author	**Conditions on space E**	**Conditions on cover $\mathcal{U}$**	**Result**
Weil	$E \times E \times [0,1]$ is normal	topologically simple;	$N \simeq E$
McCord	$E \times E \times [0,1]$ is normal	point-finite, open, the intersection of any finite subcollection of $\mathcal{U}$ is homotopically trivial;	$N \simeq E$
Borsuk	finite-dimensional, compact	regular, closed;	$N \simeq E$
Edelsbrunner and Harer	none	finite collection of sets, closed, convex, in Eucl. space	$N \simeq E$

2 General Definitions

Definition 2 (Notation)**.** *We will let $\cong$ denote isomorphism or homeomorphism, where appropriate. When discussing abstract simplicial complexes, the simplex $\{i, j\}$ may be written as ij for brevity.*

Definition 3. *The nerve of an open covering is an abstract simplicial complex defined as follows. Given an index set I and open sets $U_i \subseteq X$, we say that a finite set $J \subseteq I$ belongs to $N(\mathcal{U})$ if and only if $\bigcap_{j \in J} U_j \neq \emptyset$.*

Thus the nerve is a subset of the power set of I.

Definition 4. *A topological space X is called normal (T_4) if every two disjoint closed sets of X can be contained in disjoint open neighborhoods.*

Which types of spaces are not guaranteed to be normal? The product of two normal spaces is not necessarily normal[1]. In addition, a subset of a normal space might not be normal (i.e. not every normal Hausdorff space is a completely normal Hausdorff space). The product of uncountably many non-compact metric spaces is never normal.[2]

[1] R. Sorgenfrey
[2] A. H. Stone

Definition 5 (Weil)**.** *A space V is* paracompact *if V admits a locally finite cover by "maps", that is, by open partitions each provided with a diffeomorphism over an open subset of $\mathbb{R}^n$.*

Paracompact spaces are fairly common; all CW complexes[3], all metric spaces, and all compact spaces are paracompact. There exist spaces that are locally compact but not paracompact (e.g. the "long line"). That is, paracompactness is weaker than compactness. One important feature of paracompact Hausdorff spaces is that they are normal and admit partitions of unity, subordinate to any open cover. (See the following two definitions.)

Definition 6. *A* partition of unity *of a space X is a set R of continuous functions $\varphi : X \to [0, 1]$ such that*

1. *for every $x \in X$, there is a neighborhood of x over which all but finitely many φ are identically 0, and*
2. *for every $x \in X$, we have $\sum_{\varphi \in R} \varphi(x) = 1$.*

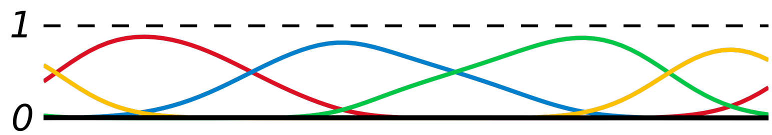

Figure: An example of a partition of unity defined on a subset of $\mathbb{R}$.

Partitions of unity can be used to extend local constructions to the whole space.

Definition 7. *Given any open cover $\{U_i\}_{i \in I}$ of a space, there exists a partition $\{\varphi_i\}_{i \in I}$ indexed over the same set I such that supp$(\varphi_i) \subseteq U_i$. Such a partition is said to be* subordinate to the open cover *$\{U_i\}_{i \in I}$.*

Every normal space admits a partition of unity that is subordinate to any given open cover of that space. More formally, let X be a normal space. If $\mathcal{U}$ is a locally finite open cover of X, then there exists a partition of unity that is precisely subordinate to $\mathcal{U}$.

Definition 8 (Weil)**.** *We say that a cover $X = (X_i)$ of E is* locally finite *if every point of E has a neighborhood which intersects only finitely many X_i. If E is locally compact, it is equivalent to say that every compact subset of E intersects only finitely many X_i.*

Definition 9 (Weil)**.** *A space B has the* extension property *if, for every closed subset A of a normal space X, every continuous mapping $A \to B$ can be extended to a continuous mapping $X \to B$.*

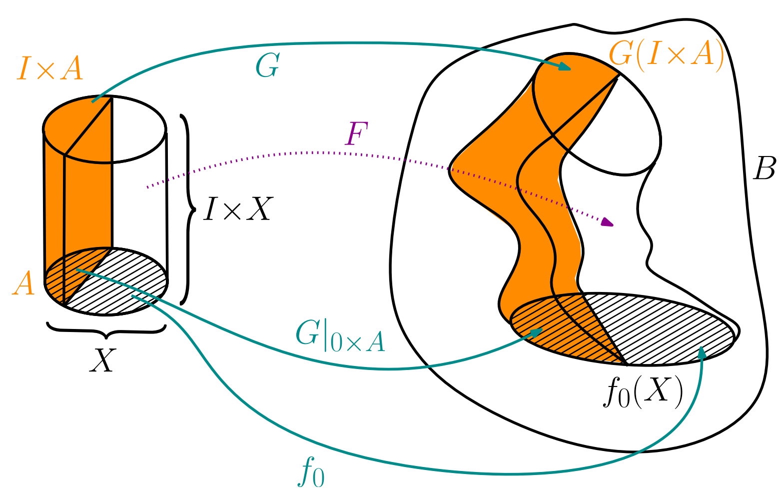

Figure: the homotopy extension property, defined in 231A.

[3] J. H. C. Whitehead

Definition 10 (231A)**.** *A space B has the* homotopy extension property *if, for every closed subset A of a normal space X, given any homotopy $G : I \times A \to B$, and given any $f_0 : X \to B$ with $f_0|_A = G|_{0\times A}$, there exists an extension $F : I \times X \to B$ such that $F|_{I\times A} = G$ and $F|_{0\times X} = f_0$.*

The homotopy extension property is a special case of the more general extension property (as in Weil's definition).

3 Weil's *Sur les theoremes de de Rham* (translated by K. Heal)

In the following sections, excerpts from the translation of the original French text are indented; personal commentary and simple examples relating to the text are unindented.

I will present Weil's proof (omitting discussion of differential forms) that any space with what we call a "topologically simple" covering is homotopy-equivalent to the nerve of that covering. More formally:

Theorem 1 (Weil)**.** *If E is a space for which $E \times E \times [0, 1]$ is normal, and if $\mathcal{U}$ is a topologically simple cover of E, the nerve $N(\mathcal{U})$ has the same homotopy type as E.*

3.1 Section 1

> **Definition 11** (Weil)**.** *Let X be a family of subsets of a space E, and a set of indices I. We say that this family is* locally finite *if every point of E has a neighborhood which intersects a finite number of X_i.*
>
> If E is locally compact, this is equivalent to saying that every compact subset of E only encounters a finite number of X_i. If $(X_i)_{i\in I}$ is a locally finite family and if $J \subset I$, we will set $X_J := \bigcap_{i\in J} X_i$.
>
> **Definition 12.** *The set N, consisting of the nonempty subsets $J \subseteq I$ such that X_J is not empty, is called the* nerve *of the family X.*
>
> **Definition 13** (Weil)**.** *Let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$, a locally finite covering of a space E by open sets (U_i). We shall say that $\mathcal{U}$ is* topologically simple *if, $\forall I \in N(\mathcal{U})$, the set U_I has the extension property.*

If $J \in N$, then J is finite; this follows from the assumption that X is locally finite.

> **The object of our study will be a smooth manifold V of dimension n, which is *paracompact*.**

One definition of "paracompact" is as follows. A *refinement* of a cover of V is another cover of V, such that every set in the new cover is a subset of some set in the old cover. The space V is then called paracompact if every open cover of V has a locally finite open refinement. As a side note, it turns out that the translation of *variété différentielle* is actually *smooth manifold*. One could make the analogy that "paracompact" is to "locally finite cover" as "compact" is to "finite cover"; clearly, the latter implies the former.

> It is equivalent to say that V admits a locally finite cover by "maps", that is, by open partitions, each provided with a diffeomorphism to an open subset of $\mathbb{R}^n$.

With this requirement that paracompact spaces be locally diffeomorphic to $\mathbb{R}^n$, Weil makes it clear that we are restricting our focus to smooth n-manifolds.

> The word "differentiable" will always be taken to mean "infinitely differentiable" (or "C^∞"). This is not really a restriction if one takes into account the Whitney theorem by which any class C^n for $n \geq 1$ admits a homeomorphism of C^n on a manifold of class C^∞. Moreover, the method that will be exposed also applies to the C class varieties for $n \geq 2$.

Given any covering of V using relatively compact open sets, by the paracompactness of V we can obtain a locally finite refinement of that covering.

> **Our main tool** will be $\mathcal{U} = (U_i)_{i \in I}$, a locally finite covering of V by relatively compact open sets U_i, which must have the following property: each non-empty set $U_J = \bigcap_{i \in J} U_i$ possesses a "differentiable retraction," that is to say a differentiable mapping $\varphi_J : U_J \times \mathbb{R} \to U_J$ such that $\varphi_J(x,t) = x$ whenever $x \in U_J$ and that $t \geq 1$, and that φ_J is constant on $U_J \times \,]-\infty, 0]$. Such a covering, equipped with the data of the retractions φ_J, will be said to be *differentially simple*.

Lemma 1. *Any space V with assumptions made previously has a differentially simple cover $\mathcal{U}$.*

Proof. We already have local finiteness, openness, and relative compactness. As of yet, however, it is not clear that there will exist such a cover whose nonempty intersections have differentiable retractions; we will show the existence of such a cover by constructing a cover with differentiable retractions φ_J.

> Let us start from a locally finite covering of V by relatively compact open maps V_i. To V_i we will then associate a diffeomorphism from V_i onto an open subset of $\mathbb{R}^n$ by means of "local coordinates", $t_1^{(i)}, \ldots, t_n^{(i)}$.

We can do this because V was assumed to be a smooth n-manifold.

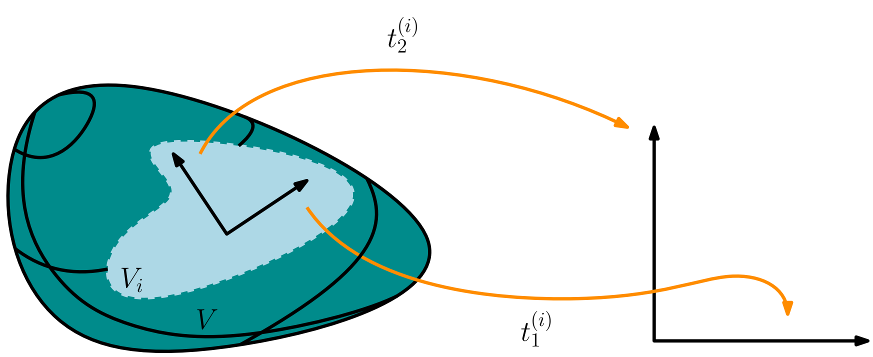

Figure: An example of such differentiable isomorphisms $t_1^{(i)}$ and $t_2^{(i)}$ from V_i into $\mathbb{R}^2$.

In the coming discussion, we will patch together a diffeomorphism $f = (f_{ij})$, using a partition (f_i) that is subordinate to (V_i). From f, we will be able to obtain the differentiable retractions φ_J that we desire. As an intermediate step to defining the maps f_{ij}, we will construct a partition (f_i) that is subordinate to (V_i). We will begin by specifying elevations of f_i throughout V_i.

> It is then possible, for each i, to define open sets W_i, W_i' and a function f_i differentiable on V so that the W_i form a cover of V, that we have $\bar{W}_i \subset W_i'$ and $\bar{W}_i' \subset V_i$, and that f_i has the value 1 on $\bar{W}_i$ and 0 outside W_i'.

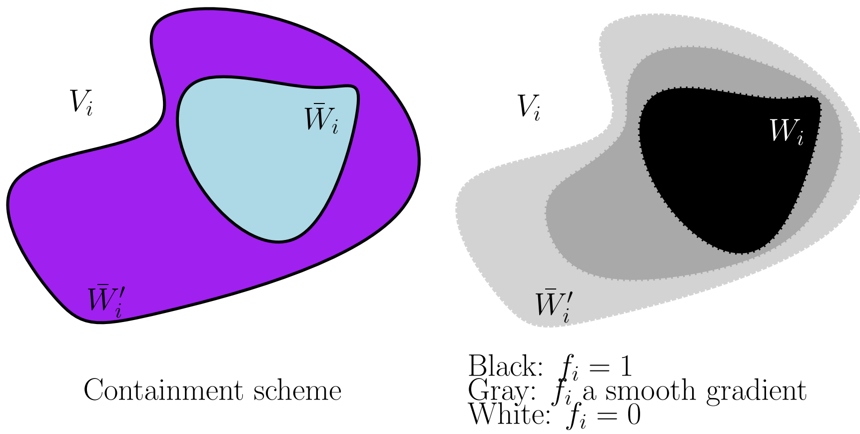

Figure: The support of local retractions f_i.

These sets W_i, W_i' exist because V is Hausdorff. Choose any $x \in V_i$. Since V_i is open by construction, the sets $V - V_i$ and $\{x\}$ are both closed in V. The space is Hausdorff, so each set is contained in an open set, and these open sets are disjoint. Furthermore, W_i can be made small enough that $\bar{W}_i \subset W_i'$.

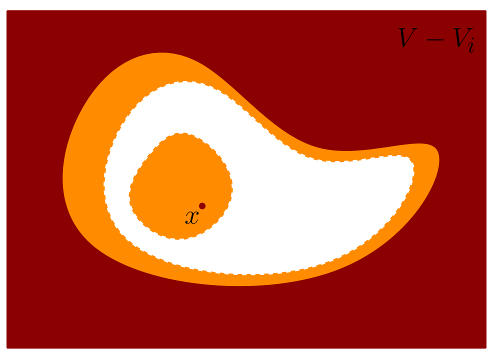

Figure: Why W_i, W_i' exist.

> Let $f_{i0} := f_i$, and let f_{ij} the function equal to $f_i t_j^{(i)}$ in V_i and 0 outside V_i. The set of functions f_{ij} for $0 \leq j \leq n$, and for all values of i, determines a mapping of V in the space $\mathbb{R}^{(A)}$, where A is the set of the pairs (i, j).

These maps f_i could be rescaled to form a partition of unity over V, but that will not be necessary in this proof.

> In this way, we designate the vector space of the mappings $A \to \mathbb{R}$ which take the value 0 everywhere except on finitely many elements of A. Moreover, the mapping $(f_{ij}) : V \to \mathbb{R}^{(A)}$ determines on each relatively compact open subset $Z \subseteq V$, a differentiable isomorphism of Z onto a submanifold of a finite-dimensional vector subspace of $\mathbb{R}^{(A)}$. We can therefore simplify the language by identifying V with its image in $\mathbb{R}^{(A)}$. On $\mathbb{R}^{(A)}$, we will put a metric space structure ("pre-Hilbert") by means of the distance $d(x, y) = [\sum_{i,j}(x_{ij} - y_{ij})^2]^{1/2}$. This makes any finite-dimensional subspace of $\mathbb{R}^{(A)}$ a Euclidean space. From the above, the distance from $\bar{W}_i$ to $V - W_i'$ is ≥ 1, since the coordinate $x_{i0} \in \bar{W}_i$ has the value 1 and $y_{i0} \in V - W_i'$ has the value 0.

Really, we can represent these maps f_{ij} in terms of a matrix. Each column corresponds to a coordinate axis in $\mathbb{R}^n$, and each row corresponds to a patch V_i of the manifold V. The distance metric making the space pre-Hilbert can be interpreted as the Frobenius norm. This matrix is guaranteed to be finite dimensional (i.e. n is finite-dimensional in the figure below) if V is compact.

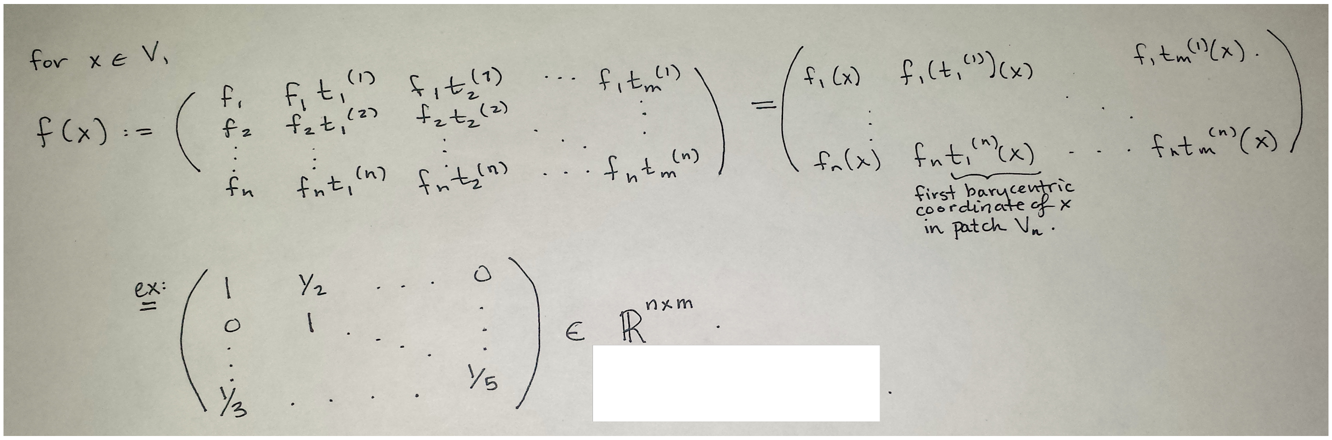

Figure: The construction of f.

> For every $x \in V$, denote by T_x the linear manifold that is tangent to V at x,and by P_x the orthogonal projection of $\mathbb{R}^{(A)}$ on T_x, considered as a linear map $P_x : \mathbb{R}^{(A)} \to T_x$. Let $U(x,r)$ denote the intersection of V with the open ball of center x and radius r; If $x \in \bar{W}_i$ and $r < 1$, then $U(x,r) \subset W_i'$.

This last inclusion holds because $d(\bar{W}_i, V - W_i') \geq 1$, as was previously shown. Recall that if V is an n-manifold, then for each $x \in V$, the tangent space T_x is an n-manifold.

> Therefore $U(x,r)$ is relatively compact provided that $r < 1$. Let $x \in \bar{W}_i$. Let E be a finite-dimensional vector space containing $\bar{W}_i'$. Taking in E orthogonal coordinates of origin x, the first n coordinate vectors being chosen in T_x, we see that x possesses an open neighborhood U contained in W_i' and having the following properties:
>
> (a) in U, P_y induces on U a differentiable isomorphism (that is, a bijective mapping, everywhere of rank n) from U to its image $U_y := P_y(U) \subseteq T_y$;
>
> (b) whenever $y, z_1, z_2 \in \bar{U}$, we have $d(z_1, z_2) < 2d(P_y(z_1), P_y(z_2))$;
>
> (c) whenever $z_0 \in U$, $d(z_0, z)^2$ is a convex function of $P_y(z)$ in U_y.
>
> In effect, this last condition means that the matrix of the second derivatives of $d(z_0, z)^2$ with respect to the coordinates of $P_y(z)$ in T_y is the matrix of a positive definite quadratic form. Now that U is small enough, this matrix is as close as we want to its value for $y = z_0 = z = x$, a value which is none other than the matrix unity.

> Let K be a compact subset of V that is covered by finitely many sets U_α having the properties (a), (b), (c). Let $0 < r(K) < 1$ be such that $U(x, r(K))$ is contained in one of the U_α for all $x \in K$. Thus $\mathcal{U}$ will have the properties (a), (b), (c) for any $x \in K$. Moreover, for $x \in K$ and $r = r(K)$, the projection $P_x[U(x,r)]$ of $U(x,r)$ onto T_x will contain all points of T_x at a distance $< r/2$ of x. Indeed, if z' is a boundary point of this projection, z' will be the point boundary of points $z'_v = P_x(z_v)$ with $z_v \in U(x,r)$. $U(x,r)$ being relatively compact on V, we can replace the z_v by a partial sequence having a limit z on V. Since P_x is a differentiable isomorphism of $U(x,r)$ onto its image, every interior point of $U(x,r)$ is projected onto an interior point of $P_x[U(x,r)]$. Hence z is a boundary point of $U(x,r)$, and $d(x,z) = r$, of or $d(x,z') > r/2$ under (b).
>
> **Let us show now** that, if $x \in K$, $0 < r \leq r(K)/4$, and $y \in U(x,r)$, then P_x induces on $U(y,r)$ a differentiable isomorphism of $U(y,r)$ over a convex subset of T_x. Since $U(y,r) \subset U(x,2r)$, the only point to prove is the convexity of $P_x[U(y,r)]$. Now it is the set of points $z' = P_x(z)$ for $z \in U(x, r(K))$ and $d(y,z)^2 < r^2$. Consider two such points $z'_1 = P_x(z_1)$, $z'_2 = P_x(z_2)$; We have for $h = 1$ and $h = 2$, $d(z_h, x) < 2r$, hence $d(z', h)/2$ whatever z' on the line segment which joins z'_1 and z'_2 in T_x. This segment is therefore contained in $P_x[U(x, r(K))]$. Since $d(y,z)^2$ is a convex function of $z' = P_x(z)$ in this last set, it is a convex function of z' on the segment that joins z'_1 and z'_2. The value of this function being $< r^2$ at the endpoints of the segment, it is also on the whole segment. This is therefore well contained in $P_x[U(y,r)]$.
>
> Let us choose for each i finitely many points $x_{i\lambda}$ from $\bar{W}_i$, such that the sets $U_{i\lambda} = U(x_{i\lambda}, r(\bar{W}'_i)/4)$ form a cover of $\bar{W}_i$.

Within the setting of $\bar{W}_i$, we will be indexing via λ.

> We say that the $U_{i\lambda}$ form a differentially simple covering of V. Since we have $x_{i\lambda} \in \bar{W}_i$ and $r(\bar{W}'_i)/4 < 1$, we have $U_{i\lambda} \subset W'_i$. **Then the $U_{i\lambda}$ are relatively compact and form a locally finite covering of** V. Let x be a common point of the (necessarily finitely many) sets $U_{i\lambda}, U_{j\mu}, U_{k\nu} \ldots$. Let r be the largest of the numbers $r(\bar{W}'_i), r(\bar{W}'_j), r(\bar{W}'_k), \ldots$. Suppose, for example, that $r = r(\bar{W}'_i)$.

We define the covering using the largest possible r in order to ensure that the entire space is covered; this may result in substantial overlap among the sets in the cover.

> Then each of the sets $U_{i\lambda}, U_{j\mu}, \ldots$ is of the form $U(y, r')$, with $y \in U(x, r')$ and $r' \leq r(\bar{W}'_i)/4$. They are all contained in $U(x, r(\bar{W}'_i))$ and, since $x \in \bar{W}'_i$, P_x induces on $U(x, r(\bar{W}'_i))$ a differentiable isomorphism in which each of the $U_{i\lambda}, U_{j\mu}, \ldots$ has as its image a convex open subset of T_x from what we have proved above. P_x also induces on their intersection a diffeomorphism over a convex open subset U' of T_x. It admits the retraction $(z', t) \to x + \lambda(t)(z' - x)$, where $\lambda(t)$ is a differentiable function on $\mathbb{R}$, equal to 0 for $t \leq 0$ and to 1 for $t \geq 1$. By virtue of the isomorphism induced by P_x, this retraction is transmitted to the intersection of the $U_{i\lambda}, U_{j\mu}, \ldots$, which completes the proof.

How do we know that the $\bigcup_{i,\lambda} U_{i,\lambda}$ will also cover the $V_i - \bar{W}_i$? By construction, the W_i also form an open cover for V. Therefore, it is enough to cover $\bigcup_i \bar{W}_i \supset \bigcup_i W_i$ in order to cover V.

Thus we have shown existence of the differentiable retractions φ_J that we needed, and we can conclude that there does indeed exist a differentially simple cover $\mathcal{U}$ for V. $\square$

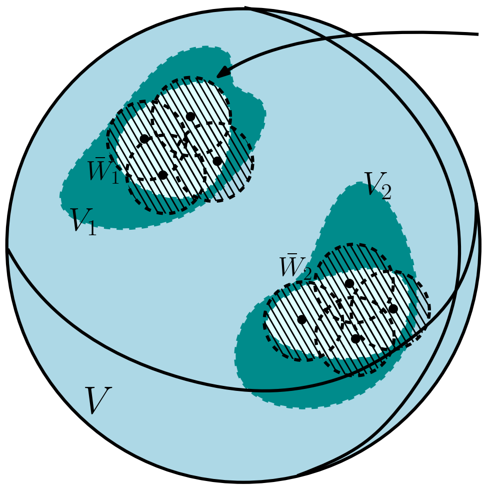

Figure: Construction of a locally finite covering of V.

> We have only actually used the fact that, when V is immersed in $\mathbb{R}^{(A)}$, every compact subset of V has bounded curvature, or else every point of V has a neighborhood which can be represented parametrically by means of functions of class C^1 whose derivatives of order 1 have their values bounded. Already for a manifold V of class C^1, it does not seem easy to construct a simple covering without first defining on V a class structure C^2 by means of the theorem of Whitney already quoted. And the problem of the *existence* of a simple cover remains open with respect to class C^0 varieties. Of course, for such a manifold, we would no longer require the retractions φ_J to be continuous. On the other hand, any locally finite simplicial complex trivially admits such a covering, formed by open stars of its vertices. In view of what follows, let us briefly recall some definitions relating to these complexes.

Next, we will construct a simple cover of, and define the geometric realization of, the nerve.

> **Definition 14.** *By an abstract simplicial complex, we mean a set N of finite nonempty subsets of any set I such that, if $J \in N$, any non-empty subset of J also belongs to N.*

That is, a simplicial complex has as simplices all of its faces.

> **Definition 15.** *We identify the abstract complex N with its geometric realization, that is to say, with the set of points $x = (x_i)_{i \in I}$ of the space $\mathbb{R}^{(I)}$ such that $\sum_{i \in I} x_i = 1, x_i \geq 0$ for all i, and that the set of $i \in I$ such that $x_i \neq 0$ belongs to N.*
>
> Without losing generality, we can assume that I is a union of sets of N (otherwise we would replace I by this union). For each i, hold the point of $\mathbb{R}^{(I)}$ whose coordinate of indices i to be 1 and the others to be zero.
>
> **Definition 16.** *The elements i of I, and also the points e_i which correspond to them, will be called the vertices of N.*
>
> **Definition 17.** *To each $J \in N$, we will assign the simplex Σ_J, i.e. the set of points $x = (x_i)$ of N, such that $x_i = 0$ for $i \notin J$.*

Lemma 2. *There is a simple cover of the nerve of $\mathcal{U}$.*

> We will call the open star St_J, i.e. the set of points $x = (x_i)$ of N such that $x_i > 0$ for $i \in J$. If $J = \{i\}$, Σ_J reduces to the vertex e_i of N, and St_J, which will be written St_i, is called the open star of e_i. For $J \in N$, we have $St_J = \bigcap_{i \in J} St_i$.
>
> If J has m elements, then if Σ_J is of dimension $m-1$, the center of gravity (or barycenter) of Σ_J will be the point $e_J = (x_i)$, with $x_i = 1/m$ for $i \in J$, and $x_j = 0$ for $j \notin J$. If the function $\lambda(t)$ is defined as above, $(x, t) \to e_J + \lambda(t)(x - e_J)$ is a retraction of St_J. The St_i therefore form a simple cover of N.

□

We now have (1) a differentially simple cover $\mathcal{U}$ of V; and (2) a simple cover of N of $\mathcal{U}$.

3.2 Section 5 (Main Statement)

> As has been remarked, the fact that the nerve N of U has the same homology as V depends only on the homological properties of the sets U_j. If we take into account the fact that they are homotopically trivial, we obtain a much more precise result: that N has the same homotopy type as V. It follows that N can be substituted for V in any problem which depends only on the homotopy type, and for example in most questions concerning fiber spaces of base V.

> Under such circumstances the nerve of a simple covering of V can therefore often be used for the same purposes as a triangulation of V. It seems that we have a very handy elementary tool in the study of varieties. This is also shown by the recent application of G. de Rham to the study of so-called torsion invariants. It is remarkable that these are not the invariants of the homotopy type. It is possible, therefore, that the nerves of the simple coverings have properties even more precise than that which will now be indicated.
>
> The following result is purely topological in nature. Let us recall that a space B has *the extension property* if every continuous mapping in B to a closed subset X of a normal space A can be extended to a continuous mapping of A into B. Then U is a locally finite covering of a space E by open sets U; let N be its nerve. We shall say that U is *topologically simple* if, for all $J \in N$, the set U_J has the extension property.

We reiterate the main result:

Theorem 2 (Weil)**.** *If E is a space for which $E \times E \times [0, 1]$ is normal, and if $\mathcal{U}$ is a topologically simple cover of E, the nerve $N(\mathcal{U})$ has the same homotopy type as E.*

Of course, $E \simeq N(\mathcal{U})$ implies $H_i(E) \cong H_i(N(\mathcal{U}))$ for every i. A simple example to show isomorphism between homology groups is $E = [0, 1]$ and $\mathcal{U} = \{U_i\}_{i=0,\ldots,k}$ for some fixed k, where $U_i = \left(\frac{i-1}{k}, \frac{i+1}{k}\right)$. Then the U_i intersect pairwise, and the nerve in abstract simplicial complex form is $N(\mathcal{U}) = \{0, \ldots, k, 01, 12, \ldots, (k-1)k\}$. Realized as a simplex, this is of course a 1-chain that is *not* a 1-cycle; therefore the reduced homology of $N(\mathcal{U})$ is trivial.

3.3 Section 5 (Proof of Main Statement)

For this proof, we associate N with its realization $|N|$, and may use these terms interchangeably. To show that E and N are homotopy equivalent, we will show the existence of two continuous maps $f : E \to N$ and $g : N \to E$ for which $f \circ g \simeq \mathrm{id}_N$ and $g \circ f \simeq \mathrm{id}_E$. This will be done in four steps.

Step 1. Showing the existence of $f : E \to N$.

> First let $\mathcal{U} = (U_i)$ be any locally finite cover of a normal space E by open sets U_i. Then there exists a partition $f = (f_i)$ subordinate to the open cover $\mathcal{U}$. We will set, for $p \in E$, $f(p) := (f_i(p))$; this f is a continuous mapping from E into the nerve N of $\mathcal{U}$, realized geometrically in accordance with the definitions recalled at the end of 1.

First, I will explain why E can be assumed to be normal in the first sentence of the proof. The space E is closed relative to $E \times E \times [0,1]$. Furthermore, a closed subspace of a normal (and Hausdorff) space is normal, so $E \times E \times [0,1]$ normal implies that E is normal. With this, we have defined a continuous map $f : E \to N$, such that $f : p \mapsto (f_i(p))$.

Step 2. Showing the existence of $g : N \to E$.

> If $p \in E$, and if J is the set of $i \in I$ such that $p \in U_i$, $f(p)$ is in the simplex Σ_J of N.

Weil is saying that in this case, p is covered by U_i for all $i \in J$, and therefore is covered by U_J. This implies an overlap of the covers in J, so this $J \in N$ as an abstract simplicial complex. Every cardinality-n element (set) of an abstract simplicial complex corresponds to an n-simplex in the realized complex. That is, once the nerve is realized, J is assigned to the simplex Σ_J.

> If (f_i') is another partition subordinate to $\mathcal{U}$, the line segment joining $f(p)$ and $f'(p)$ is contained in Σ_J, and is therefore in N. Consequently, the mapping $p \to (1-t)f(p)+tf'(p)$ is a homotopy joining f to f'; the homotopy class of f is thus completely determined by the data of $\mathcal{U}$.
>
> Suppose now that $U_J := \bigcap_{i \in J} U_i$, for $J \in N$, have all trivial homotopy groups. In other words, any continuous mapping into one of the U_j of the boundary of a simplex of dimension m can be extended to all the simplex. For $m = 1$, this means that U_J is path-connected.
>
> For all $J \in N$, let e_J be the center of gravity of Σ_J. Consider an increasing sequence $J_0 \subset J_1 \subset \cdots \subset J_m$ of distinct elements of N.

These elements are ordered by set inclusion. Since N is not totally ordered (e.g. $12 \notin 13$ and $13 \notin 12$), not every element of N will fit into the same sequence; that is, m will in general not be the cardinality of N. We will have several increasing sequences of that form. For example, if $N = \{1, 2, 3, 4, 12, 13, 23, 123\}$, then we will have the following sequences:

$$\{1, 12, 123\}, \{1, 13, 123\}, \{2, 12, 123\}, \{3, 13, 123\}, \{2, 23, 123\}, \{3, 23, 123\},$$

$$\{1, 12\}, \{1, 13\}, \{2, 12\}, \{3, 13\}, \{2, 23\}, \{3, 23\}, \{1\}, \{2\}, \{3\}, \{4\}.$$

> For such a sequence, let $\Sigma'(J_0, \ldots, J_m)$ be the simplex of vertices $e_{J_0}, \ldots, e_{J_m}$. N is the union of all these simplexes, which form the barycentric subdivision.

Using the same example as before, a simplex might look like:

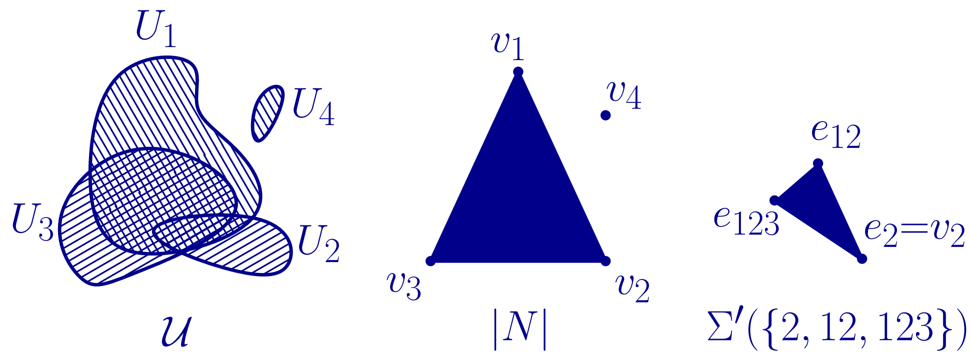

Figure: Forming one simplex in the barycentric subdivision the nerve of the cover $\mathcal{U}$.

We can see that, indeed, the union of all such simplices will form the barycentric subdivision of the complex in the example.

> We will define by recursion a continuous mapping $g : N \to E$ such that $g(\Sigma'(J_0, \ldots, J_m)) \subset U_{J_0}$ for every sequence $J_0, \ldots, J_m$. This map sends $g(e_J) \to U_J$ for all $J \in N$.

That is,

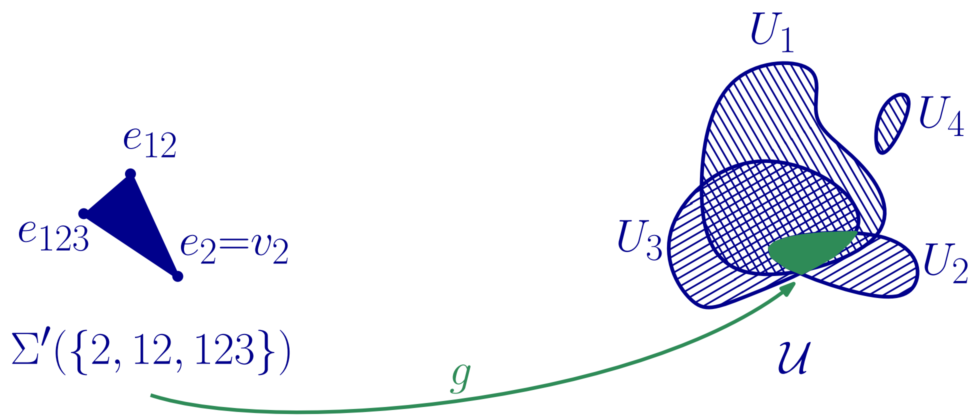

Figure: The action of g on N.

> Suppose that g is defined on the simplexes of the barycentric subdivision of N of dimension $\leq m-1$. Then g is defined on the boundary of the simplex $\Sigma'(J_0, \ldots, J_m)$ which is the union of simplices $\Sigma'_\mu := \Sigma'(J_0, \ldots, J_{\mu-1}, J_{\mu+1}, J_m)$ for $0 \leq \mu \leq m$. According to the recursive hypothesis, we have $g(\Sigma'_0) \subset U_{J_1} \subset U_{J_0}$, and $g(\Sigma'_\mu) \subset U_{J_0}$ for $1 \leq \mu \leq m$. Then g can be extended to a mapping from $\Sigma'(J_0, \ldots, J_m)$ to U_{J_0}.
>
> If, moreover, g' is another mapping from N to E that satisfies the same condition, we can, by a quite similar recursion, construct a homotopy joining g to g'. The homotopy class of g is therefore determined by the condition that has been imposed.

Step 3. Showing that $f \circ g \simeq \mathrm{id}_N$.

> Let us show that under these conditions $f \circ g$ is a mapping from N to itself that is homotopic to the identity map in N. Let F_i be the union of the images under g of all the simplexes $\Sigma'(J_0, \ldots, J_m)$ for which $i \in J_0$. Since there are only finitely many of these simplexes, F_i is a compact subset of U_i. For each i, let U'_i be an open subset of U_i containing F_i, such that $\bar{U}'_i \subset U_i$, and that the U'_i still form a cover of E.

For example,

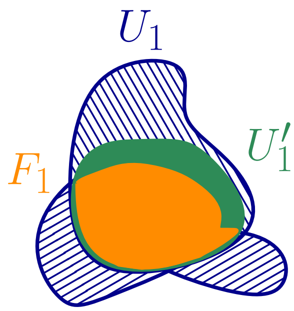

Figure: Definiting the subset U_i'.

Since the choice of the partition (f_i) subordinate to $\mathcal{U}$ has no influence on the homotopy class of f, we can assume it is chosen so that $f_i > 0$ on F_i and $f_i = 0$ outside U_i', for all $i \in I$.

Step 4. Showing that $g \circ f \simeq \mathrm{id}_E$.

Finally, let $p \in E$, and let J be the set of $i \in I$ such that $f_i(p) > 0$. Then we have $p \in U_i'$ for all $i \in J$. We also have $f(p) \in \Sigma_J$, and therefore $f(p)$ will belong to a simplex of the barycentric subdivision of Σ_J. But these are, with the notations used above, the simplexes $\Sigma'(J_0, \ldots, J_m)$ with $J_m \subset J$. If we then take $i \in J_0$, we will have $g(f(p)) \in F_i$. Therefore p and $g(f(p))$ are both in U_i'. Now let $X_i := \bar{U}_i'$, and let N' be the nerve of the family (X_i). We will have $N' \subset N$. If we suppose now that the U_J have the extension property, that is to say that $\mathcal{U}$ is topologically simple, we see that the families $(X_i)_{i \in I}$ and $(U_J)_{J \in N'}$ satisfy all the conditions of the lemma in this section. From the corollary of this lemma, we can then assert that $g \circ f$ is homotopic to the identity mapping of E provided that $E \times E \times [0,1]$ is normal.

□

We present the following lemma and corollary without proof.

Lemma 3. *Let E be a space such that $E \times E \times [0,1]$ is normal. Let $(X_i)_{i \in I}$ be a locally finite family of closed subsets of E. Let N be its nerve; for $J \in N$, let $X_J := \bigcap_{i \in J} X_i$. Let $(U_J)_{J \in N}$ be a family of parts of E such that for all $J \in N$, U_J has the extension property and contains X_J, and that we have $U_J \subset U_{J'}$ whenever $J \supset J'$, $J \in N$, $J' \in N$.*
Then there is a continuous mapping $F(x,y,t)$ of $\bigcup_{i \in I}(X_i \times X_i \times [0,1])$ to E such that for all $J \in N$, $x \in X_J$ and $y \in X_J$ we have $F(x,x,t) = x$ for all t, $F(x,y,0) = x$, and $F(x,y,1) = y$.

Corollary 1. *The hypotheses being those of the lemma, let f and f' be two continuous mappings of a space A into E such that, for any $u \in A$, there exists an $i \in I$ for which $f(u) \in X_i$ and $f'(u) \in X_i$. Then f and f' are homotopic.*

That is, $F(f(u), f'(u), t)$ as defined in the lemma serves as a homotopy between f and f'.

4 McCord

Definition 18. *A space X is* homotopically trivial *if $\pi_i(X, x) = 0$ for all $i \geq 0$.*

Definition 19. *The open cover $\mathcal{U}$ will be called* basis-like *if the intersection of any two members of $\mathcal{U}$ is a union of members of $\mathcal{U}$. This is equivalent to saying that $\mathcal{U}$ is a basis for a topology on X smaller than the given one.*

Definition 20. *An open cover $\mathcal{U}$ is* point-finite *if each point of X is contained in only finitely many members of $\mathcal{U}$.*

Definition 21. *A map $f : X \to Y$ is a* weak homotopy equivalence *if the induced maps $f_* : \pi_i(X, x) \to \pi_i(Y, fx)$ are isomorphisms for all $x \in X$ and all $i \geq 0$.*

Theorem 3. *Let X be a space and let $\mathcal{U}$ be a point-finite, basis-like, open cover of X by homotopically trivial sets. Then there exists a weak homotopy equivalence $f : |K(\mathcal{U})| \to X$.*

Suppose $\mathcal{U}$ is an open cover (i.e. a collection of nonempty open subsets of X whose union is X) of a space X. Let $K(\mathcal{U})$ be the complex whose vertices are the members of $\mathcal{U}$ and whose simplices are the finite totally ordered subcollections of $\mathcal{U}$ (where $\mathcal{U}$ is partially ordered by inclusion). This construction of $K(\mathcal{U})$ is reminiscent of Weil's sequences $J_0 \subset \cdots \subset J_m$ that form $\Sigma'(J_0, \ldots, J_m)$. We see that $K(\mathcal{U})$ is a subcomplex of $N(\mathcal{U})$.

Theorem 4 (McCord's interpretation of Weil's Theorem)**.** *Let E be a space for which $E \times E \times [0, 1]$ is normal. Let $\mathcal{U}$ be a point-finite (and locally finite) open cover of E such that the intersection of any finite subcollection of $\mathcal{U}$ is homotopically trivial (and solid). Then there exists a homotopy equivalence $|N(\mathcal{U})| \to E$.*

[How this compares to Weil.] The conditions asserted in Theorem 4 imply those from Theorem 1, to achieve an identical result. McCord's assumptions that $\mathcal{U}$ be locally finite and that $E \times E \times [0, 1]$ be normal are required in order to use Lemma 1 from Weil's proof. Furthermore, one could take the closure $\bar{\mathcal{U}}$ of the covering (i.e. the closure of each set in the covering). The following lemma will complete this argument.

Lemma 4. *If $\mathcal{U}$ is point-finite, locally finite, and with finite intersections being homotopically trivial, then $\mathcal{U}$ is topologically simple.*

Proof. If $\mathcal{U}$ is point finite and locally finite, then *each* of its intersections involve only finitely many sets. Therefore, every intersection in the nerve N involves only finitely many sets. Therefore, every simplex $J \in N$ is homotopically trivial. Since any homotopically trivial space has the extension property (one could use a linear homotopy), $\mathcal{U}$ is topologically simple. ☐

Although Theorem 4 does seem more intuitive (and also allows him to recast Weil's result in the context of his own proof), Weil achieves the same result with less restrictive conditions.

With this said, McCord's primary contribution (Theorem 5) is a slightly weaker result than Weil's, but with the benefit of much less restrictive conditions than those previously stated. In particular, Theorem 5 does not rely on an assumption that any space s are normal.

Theorem 5 (McCord's weaker version of Weil's theorem)**.** *Let X be a space and let $\mathcal{V}$ be a point-finite open cover of X such that the intersection of any (finite) subcollection of $\mathcal{V}$ is homotopically trivial. Then there exists a* ***weak*** *homotopy equivalence $|N(\mathcal{V})| \to X$.*

If our goal is solely to identify the homology groups of X up to isomorphism, weak homotopy equivalence is good enough, since a weak homotopy equivalence induces isomorphisms on all homology and cohomology groups[4].

[4]A. Hatcher, Proposition 4.21.

5 Borsuk

Assumption: E is a finite-dimensional compactum. Decompose E into a finite sum of closed sets. What happens when E has a decomposition into a finite sum of absolute retracts such that every nonempty intersection of those retracts is also an absolute retract?

Definition 22. *A topological space X is called regular (T_3) if every closed subset E of X and a point p not contained in E admit non-overlapping open neighborhoods.*

Definition 23. *A compactum A is an absolute retract whenever a topological image of A in any space X is necessarily a retract of X.*

Definition 24. *A decomposition of a space X is a cover $(U_i)_{i\in I}$ of closed sets for which $\bigcup_{i\in I} U_i \supseteq X$. A regular decomposition is a decomposition into regular covers.*

Definition 25 (Munkres)**.** *A If the abstract simplicial complex X is isomorphic to the vertex scheme of the simplicial complex K, then we call K a geometric realization of X.*

Definition 26. *A polytope is a set P such that there exists a simplicial complex K with realization $|K| = P$.*

Theorem 6 (Corollary 3)**.** *Let the simplicial complex N be a geometric realization of the nerve of a regular decomposition of a finite-dimensional compactum E. Then E and the polytope $|N|$ are homotopy equivalent.*

[**How this compares to McCord.**] If the compactum E is also Hausdorff, then it is compact and normal; by Lemma 5, $E\times E\times[0,1]$ is also normal. Furthermore, since E is a finite-dimensional compactum, we can find a regular decomposition (in particular, of convex sets) that has finitely many elements. Thus we can assume the decomposition $\mathcal{U}$ is point-finite and locally finite, and with homotopically trivial intersections. Therefore, Borsuk's assumptions imply McCord's.

Lemma 5. *If the space E is compact Hausdorff, then the space $E \times E \times [0,1]$ is normal with respect to the product topology.*

The product of two compact spaces is compact[5], and the product of two Hausdorff spaces is Hausdorff. Furthermore, all compact spaces are paracompact, and all paracompact Hausdorff spaces are normal. It follows that the product of two compact Hausdorff spaces is normal. From this discussion we see that: if E is compact Hausdorff, then $E \times E \times [0,1]$ is normal. □

6 Edelsbrunner and Harer

The following theorem is due to Leray, but is stated in a simpler (Euclidean) form for application by Edelsbrunner and Harer.

Theorem 7. *(Leray) If $\mathcal{U}$ is a finite collection of closed, convex sets in Euclidean space, then the nerve of $\mathcal{U}$ is homotopy equivalent to the union of the sets in $\mathcal{U}$.*

[5] A. N. Tikhonov

This theorem can be restated as: Let $\mathcal{U} = \{U_i\}$ be a finite collection of closed, homotopically trivial sets in $\mathbb{R}^n$; let N be its nerve. Then $N \simeq \bigcup_i U_i$. In application, we are usually interested in the case where $\mathcal{U}$ covers some underlying manifold E. In general, a cover $\mathcal{U}$ need not have the same homology groups as, much less be homotopy equivalent to, this underlying space; it would seem that Edelsbrunner and Harer's claim is really only half of Weil's. However, the (somewhat restrictive) condition that each U_i in the cover is convex implies that the U_i are star-convex, and it is easy to find a linear homotopy to deformation retract each covering set to E. That is, $E \simeq \mathcal{U}$, and it follows that if E is a manifold, and $\mathcal{U}$ is a finite collection of closed, convex sets in Euclidean space that covers E, then $N(\mathcal{U}) \simeq E$.

Have a great winter break! ☺

7 References

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